# Spin polarisabilities of the nucleon at NLO in the chiral expansion

K. B. Vijaya Kumar[*], Judith A. McGovern[†] and Michael C. Birse[‡]

*Theoretical Physics Group, Department of Physics and Astronomy*
*University of Manchester, Manchester, M13 9PL, U.K.*

## Abstract

We present a calculation of the fourth-order (NLO) contribution to spin-dependent Compton scattering in heavy-baryon chiral perturbation theory, and we give results for the four spin polarisabilities. No low-energy constants, except for the anomalous magnetic moments of the nucleon, enter at this order. For forward scattering the fourth-order piece of the spin polarisability of the proton turns out to be almost twice the size of the leading piece, with the opposite sign. This leads to the conclusion that no prediction can currently be made for this quantity. For backward scattering the fourth-order contribution is much smaller than the third-order piece which is dominated by the anomalous scattering, and so cannot explain the discrepancy between the CPT result and the current best experimental determination.

12.39Fe 13.60Fz 11.30Rd

[*]Permanent address: Department of Physics, University of Mangalore, Mangalore 574 199, India Electronic address: kbvijayakumar@yahoo.com

[†]Electronic address: judith.mcgovern@man.ac.uk

[‡]Electronic address: mike.birse@man.ac.uk

Compton scattering from the nucleon has recently been the subject of much work, both experimental and theoretical. For the case of unpolarised protons the experimental amplitude is well determined, and in good agreement with the results of heavy-baryon chiral perturbation theory (HBCPT). However the situation with regard to scattering from polarised targets is less satisfactory, not least because until very recently no direct measurements of polarised Compton scattering had been attempted.

The usual notation for spin-dependent pieces of the scattering amplitude in the Breit frame is, for incoming real photons of energy $\omega$ and momentum $\mathbf{q}$ to outgoing real photons of the same energy energy and momentum $\mathbf{q}'$,

$$
\begin{aligned}
T &= \epsilon'^{\mu}\Theta_{\mu\nu}\epsilon^{\nu} \\
&= i\boldsymbol{\sigma}\cdot(\boldsymbol{\epsilon}'\times\boldsymbol{\epsilon})\,A_3(\omega,\theta) + i\boldsymbol{\sigma}\cdot(\hat{\mathbf{q}}'\times\hat{\mathbf{q}})\,\boldsymbol{\epsilon}'\cdot\boldsymbol{\epsilon}\,A_4(\omega,\theta) \\
&\quad + \Big(i\boldsymbol{\sigma}\cdot(\boldsymbol{\epsilon}'\times\hat{\mathbf{q}})\,\boldsymbol{\epsilon}\cdot\hat{\mathbf{q}}' - i\boldsymbol{\sigma}\cdot(\boldsymbol{\epsilon}\times\hat{\mathbf{q}}')\,\boldsymbol{\epsilon}'\cdot\hat{\mathbf{q}}\Big)\,A_5(\omega,\theta) \\
&\quad + \Big(i\boldsymbol{\sigma}\cdot(\boldsymbol{\epsilon}'\times\hat{\mathbf{q}}')\,\boldsymbol{\epsilon}\cdot\hat{\mathbf{q}}' - i\boldsymbol{\sigma}\cdot(\boldsymbol{\epsilon}\times\hat{\mathbf{q}})\,\boldsymbol{\epsilon}'\cdot\hat{\mathbf{q}}\Big)\,A_6(\omega,\theta) \\
&\quad + i\boldsymbol{\sigma}\cdot(\hat{\mathbf{q}}'\times\hat{\mathbf{q}})\,\boldsymbol{\epsilon}'\cdot\hat{\mathbf{q}}\,\boldsymbol{\epsilon}\cdot\hat{\mathbf{q}}'\,A_7(\omega,\theta) + \text{spin independent pieces}
\end{aligned}
\tag{1}
$$

where hats indicate unit vectors. By crossing symmetry the functions $A_i$ are odd in $\omega$. The leading pieces in an expansion in powers of $\omega$ are given by low-energy theorems [1], and the next terms contain the spin polarisabilities $\gamma_i$:

$$
\begin{aligned}
A_3(\omega,\theta) &= \frac{e^2\omega}{2m_N^2}\Big(Q(Q+2\kappa) - (Q+\kappa)^2\cos\theta\Big) + 4\pi\omega^3(\gamma_1+\gamma_5\cos\theta) \\
&\quad - \frac{e^2 Q(Q+2\kappa)\omega^3}{8m_N^4} + \mathcal{O}(\omega^5) \\
A_4(\omega,\theta) &= -\frac{e^2\omega}{2m_N^2}(Q+\kappa)^2 + 4\pi\omega^3\gamma_2 + \mathcal{O}(\omega^5) \\
A_5(\omega,\theta) &= \frac{e^2\omega}{2m_N^2}(Q+\kappa)^2 + 4\pi\omega^3\gamma_4 + \mathcal{O}(\omega^5) \\
A_6(\omega,\theta) &= -\frac{e^2\omega}{2m_N^2}Q(Q+\kappa) + 4\pi\omega^3\gamma_3 + \mathcal{O}(\omega^5) \\
A_7(\omega,\theta) &= \mathcal{O}(\omega^5)
\end{aligned}
\tag{2}
$$

where the charge of nucleon is $Q = (1+\tau_3)/2$ and its anomalous magnetic moment is $\kappa = (\kappa_s + \kappa_v\tau_3)/2$. Only four of the polarisabilities are independent since three are related by $\gamma_5 + \gamma_2 + 2\gamma_4 = 0$. The polarisabilities are also isospin dependent.

None of the polarisabilities has yet been measured directly. The best estimates that exist at present are for forward scattering (where only $A_3$ contributes). The quantity $4\pi f_2(0)$ is defined as $\mathrm{d}A_3(\omega,0)/\mathrm{d}\omega$ at $\omega = 0$, and according to the LET it depends only on $\kappa^2$ [1]. The relevant polarisability is $\gamma_0 = \gamma_1 + \gamma_5$, which is related via a dispersion relation to measurements at energies above the threshold for pion production, $\omega_0$:

$$
\gamma_0 = \frac{1}{4\pi^2}\int_{\omega_0}^{\infty}\frac{\sigma_-(\omega) - \sigma_+(\omega)}{\omega^3}d\omega,
\tag{3}
$$

where $\sigma_\pm$ are the parallel and antiparallel cross-sections for photon absorption; the related sum rule for the model-independent piece, due to Gerasimov, Drell and Hearn [2], has the same form except that $1/\omega$ replaces $1/\omega^3$.

Before direct data existed, the relevant cross-sections were estimated from multipole analyses of pion electroproduction experiments [3,4]. These showed significant discrepancies between the LET and the GDH sum rule for the difference of $f_2(0)$ for the proton and neutron, though the sum was in good agreement. Indeed even the sign of the difference was different. More recently, measurements have been made with MAMI at Mainz, for photon energies between 200 and 800 MeV; the range will be extended downward to 140 MeV, and a future experiment at Bonn will extend it upwards to 3 GeV [5]. The preliminary data from MAMI [5] suggest a continuing discrepancy between the LET and the sum rule for the proton, though a smaller one than given by the multipole analysis. The most recent analysis using electroproduction data, which pays particular attention to the threshold region, also reduces the discrepancy somewhat [6].

The MAMI data does not currently go low enough in energy to give a reliable result for the spin polarisability, $\gamma_0$. However electroproduction data have also been used to extract this quantity; Sandorfi *et al.* [4] find $\gamma_0^p = -1.3 \times 10^{-4}\,\mathrm{fm}^4$ and $\gamma_0^n = -0.4 \times 10^{-4}\,\mathrm{fm}^4$, while the more recent analysis of Drechsel *et al.* [7] gives a rather smaller value of $\gamma_0^p = -0.6 \times 10^{-4}\,\mathrm{fm}^4$. (We shall use units of $10^{-4}\,\mathrm{fm}^4$ for polarisabilities from now on.)

The spin polarisability has also been calculated in the framework of HBCPT: at lowest (third) order in the chiral expansion this gives $\gamma_0 = \alpha_{em}g_A^2/(24\pi^2 f_\pi^2 m_\pi^2) = 4.51$ for both proton and neutron, where the entire contribution comes from $\pi N$ loops. The effect of the $\Delta$ enters in counter-terms at fifth order in standard HBCPT, and has been estimated to be so large as to change the sign [8]. The calculation has also been done in an extension of HBCPT with an explicit $\Delta$ by Hemmert *et al.* [9]. They find that the principal effect is from the $\Delta$ pole, which contributes $-2.4$, with the effect of $\pi\Delta$ loops being small, $-0.2$. Clearly the next most important contribution is likely to be the fourth-order $\pi N$ piece, and this is the result which is presented here. The effects of the $\Delta$ at NLO involve unknown parameters; one might hope that the loop pieces at least will be small.

One other combination of the polarisabilities has been estimated from low-energy data for Compton scattering from the proton by Tonnison *et al.* [10], namely that for backward scattering, $\gamma_\pi = \gamma_1 - \gamma_5$. This quantity is dominated by the anomalous $\pi^0$ exchange graph, which vanishes for forward scattering, but at third and fourth order there are also pion loop contributions. The experimental value is $\gamma_\pi = -27.1$ with experimental and theoretical errors of about 10% each. The HBCPT result of Hemmert *et al.* is $-36.7$, of which $-43.5$ is the anomalous contribution, $4.6$ is the $\pi N$ piece and $2.2$ comes from including the $\Delta$ [9].

In the current paper we calculate the fourth-order contributions to all four polarisabilities. Results for $\gamma_0$ have been previously presented by Ji and collaborators [11] and by ourselves [12].

To calculate the spin-dependent scattering amplitude, we work in the gauge $A_0 = 0$, or in the language of HBCPT, $v \cdot \epsilon = 0$, where $v^\mu$ is the unit vector which defines the nucleon rest frame. Note that in this gauge there is no lowest-order coupling of a photon to a nucleon; the coupling comes in only at second order. The Feynman vertex consists of two pieces, one proportional to the charge current and one to the magnetic moment:
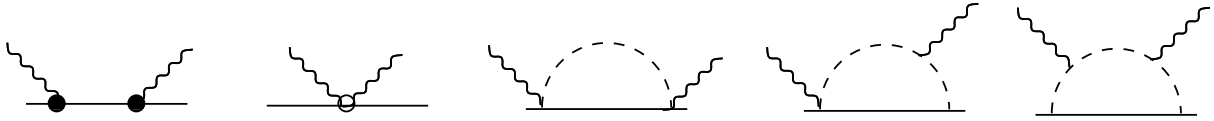
**Fig. 1:** Diagrams which contribute to spin-dependent Compton scattering in the $\epsilon \cdot v = 0$ gauge at LO. The solid dots are vertices from $\mathcal{L}^{(2)}$ and the open circle is a vertex from $\mathcal{L}^{(3)}$.

$$\frac{ie}{2M}\Big(Q\epsilon \cdot (p_1 + p_2) + 2(Q + \kappa)[S \cdot \epsilon, S \cdot q]\Big). \tag{4}$$

This and all other vertices are taken from the review of Bernard *et al.* [13].

At leading (third) order the fixed terms in the amplitudes $A_3$ to $A_6$ are reproduced, with $\kappa$ replaced by its bare value, by the combination of the Born terms and the seagull diagram, which has a fixed coefficient in the third-order Lagrangian [8]. The loop diagrams of Fig. 1 have contributions of order $\omega$ which cancel and so do not affect the LET. However they do give contributions to the polarisabilities.

At NLO, the diagrams which contribute are given in Fig. 2. There can be no seagulls at this order; since the NLO pieces of the $A_i(\omega)$ are of fourth chiral order and are odd in $\omega$, they will have expansions of the form $e^2\omega\,(a\,m_\pi + b\,\omega^2/m_\pi + \ldots)$. These non-analytic powers of $m_\pi^2$ cannot be present in the basic couplings in the Lagrangian, but can only be generated from loops. It follows that there are no undetermined low-energy constants in the final amplitude.

The insertion on the nucleon propagator of Figs. 2a, b, h and i needs some explanation. Denoting the external nucleon residual momentum by $p$, the energy $v \cdot p$ starts at second chiral order with the mass shift and kinetic energy. In contrast the space components of $p$ and all components of the loop momentum $l$ and the photon momentum are first order. The propagator with an insertion consists of both the second term in the expansion of the lowest-order propagator, $i/(v \cdot l + v \cdot p)$, in powers of $v \cdot p/v \cdot l$, and also the insertions from $\mathcal{L}^{(2)}$. The second-order mass shift and external kinetic energy cancel between the two.

To calculate the polarisabilities, it is sufficient to work in the Breit frame in which the sum of the incoming and outgoing nucleon three-momenta is zero. In the centre-of-mass frame, this is not the case, and at fourth chiral order new terms appear in the amplitudes which were absent from the general form of the amplitude given by Hemmert *et al.* [9]. They can, however, easily be distinguished from the terms in which we are interested because they appear to have the wrong crossing symmetry, contributing at order $\omega^2$ and $\omega^4$ in the expansions of the $A_i$. These terms in fact arise from boost corrections to the third-order contributions, the $\omega^2$ terms from the LET pieces, and the $\omega^4$ terms from the third-order polarisabilities. There is also one other spin-dependent term—we need to remember that a under a Lorentz boost, Wigner rotations can generate spin-dependent amplitudes from spin-independent ones. This effect is of second order in the chiral expansion, in contrast to the other terms in a boost which are of first chiral order. Since the Thomson term for spin-independent Compton scattering is of second order in HBCPT, through the Wigner rotation it gives rise to a fourth-order contribution to $A_4$ in any but the Breit frame.

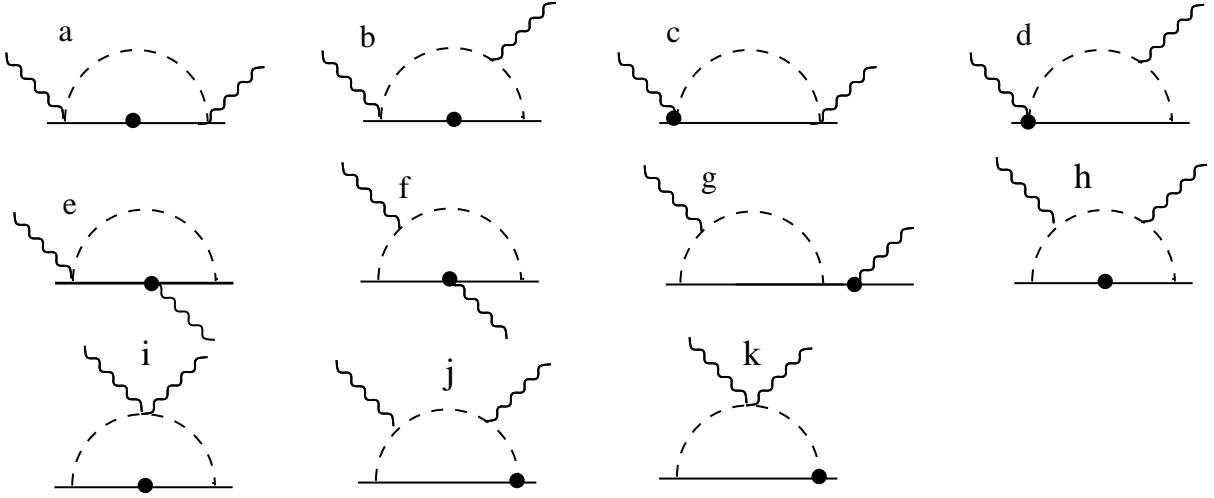Returning to the Breit frame, only diagrams 2a-h contribute. When the amplitudes are

**Fig. 2:** Diagrams which contribute to spin-dependent forward Compton scattering in the $\epsilon \cdot v = 0$ gauge at NLO. The solid dots are vertices from $\mathcal{L}^{(2)}$.

Taylor expanded, there are contributions at order $\omega$ and $\omega^3$. The former do not violate the LETs, however; they simply provide the pieces proportional to $\delta\kappa_v = -g_A^2 m_\pi M_N / 4\pi f_\pi^2$ necessary to satisfy the LET's to fourth order. The contributions of the various diagrams from Fig. 2 to the polarisabilities are given in Table 1. It can be seen that the requirement $\gamma_5 + \gamma_2 + 2\gamma_4 = 0$ is satisfied, which provides a non-trivial check on the results.

The loop contributions to the polarisabilities to NLO are then

$$\gamma_1 = \frac{\alpha_{em}g_A^2}{24\pi^2 f_\pi^2 m_\pi^2}\left[1 - \frac{\pi m_\pi}{8M_N}(8 + 5\tau_3)\right]$$

$$\gamma_2 = \frac{\alpha_{em}g_A^2}{48\pi^2 f_\pi^2 m_\pi^2}\left[1 - \frac{\pi m_\pi}{4M_N}(8 + \kappa_v + 3(1 + \kappa_s)\tau_3)\right]$$

$$\gamma_3 = \frac{\alpha_{em}g_A^2}{96\pi^2 f_\pi^2 m_\pi^2}\left[1 - \frac{\pi m_\pi}{4M_N}(6 + \tau_3)\right]$$

$$\gamma_4 = \frac{\alpha_{em}g_A^2}{96\pi^2 f_\pi^2 m_\pi^2}\left[-1 + \frac{\pi m_\pi}{4M_N}(15 + 4\kappa_v + 4(1 + \kappa_s)\tau_3)\right]$$

$$\gamma_0 = \frac{\alpha_{em}g_A^2}{24\pi^2 f_\pi^2 m_\pi^2}\left[1 - \frac{\pi m_\pi}{8M_N}(15 + 3\kappa_v + (6 + \kappa_s)\tau_3)\right] \tag{5}$$

Although the subleading pieces have a factor of $m_\pi/M_N$ compared with the leading piece, the numerical coefficients are often large. The anomalous magnetic moments are $\kappa_s = -0.12$ and $\kappa_v = 3.71$; with these values the numerical results for the polarisabilities to fourth order are

$$\gamma_1 = [-21.3] + 4.5 - (2.1 + 1.3\,\tau_3)$$
$$\gamma_2 = 2.3 - (3.1 + 0.7\,\tau_3)$$
$$\gamma_3 = [10.7] + 1.1 - (0.8 + 0.1\,\tau_3)$$

5

| Diagram | $\gamma_1$ | $\gamma_5$ | $\gamma_2$ | $\gamma_4$ | $\gamma_3$ |
|---|---|---|---|---|---|
| a | $-18$ | $-12$ | | | |
| b | $13$ | $8$ | | $4$ | |
| c | $-12\tau_3$ | | | | |
| d | $4\tau_3$ | | | | |
| e | $-3+3\tau_3$ | | | | |
| f | | $-\mu_v + \mu_s\tau_3$ | $\mu_v - \mu_s\tau_3$ | | $-(1-\tau_3)/2$ |
| g | | $-2(\mu_v + \mu_s\tau_3)$ | $-2(\mu_v + \mu_s\tau_3)$ | $2(\mu_v + \mu_s\tau_3)$ | $-1-\tau_3$ |
| h | | | $-7$ | $3/2$ | $-3/2$ |

**Table 1:** Contributions to the polarisabilities from the diagrams of Fig. 2, in units of $\alpha_{em}g_A^2/(192\pi m_N M_\pi f_\pi^2)$. The polarisabilities are defined in Eq. 2.

$$\gamma_4 = [-10.7] - 1.1 + (3.9 + 0.5\,\tau_3)$$
$$\gamma_0 = 4.5 - (6.9 + 1.5\,\tau_3)$$
$$\gamma_\pi = [-42.7] + 4.5 + (2.7 - 1.1\,\tau_3) \tag{6}$$

The term in square brackets, where it exists, is the third-order anomalous contribution. (The fourth-order anomalous contribution is proportional to the difference in the incoming and outgoing photon energies, which vanishes in the Breit frame.)

The NLO contributions are disappointingly large, and call the convergence of the expansion into question. While the fifth-order terms have also been estimated to be large [8], this is due to physics beyond pion-nucleon loops, namely the contribution of the $\Delta$. Our results show that even in the absence of the $\Delta$, convergence of HBCPT for the polarisabilities has not yet been reached. However the convergence is better for $\gamma_\pi^p$, and so our results do not shed light on the large dicrepancy between the experimental and theoretical determinations.

After we completed this paper the Jülich group released their results for the fourth-order polarisabilities [14]. They choose not to include the one-particle-reducible diagram of Fig. 2g in their definition of the polarisabilities. With the addition of that contribution, their results are in agreement with ours.

### APPENDIX A: FULL AMPLITUDE

The full amplitude in the Breit frame for diagrams 2a-2h are as follows. The notation $t_i$ is used for the tensor structures which multiply the amplitudes $A_i$.

$$T_a = \frac{g^2 e^2}{4m_N f_\pi^2}\Big[\big(m^2 - \omega^2(1 + \cos\theta)\big)\frac{\partial J_0(\omega, m^2)}{\partial\omega} - 2\omega J_0(\omega, m^2)\Big]t_3 - (\omega \to -\omega)$$

$$T_b = -\frac{g^2 e^2}{2m_N f_\pi^2}\Big[\frac{2m^2}{\omega}\big(J_2'(\omega, m^2) - J_2'(0, m^2)\big)t_3 + (1 + \cos\theta)\frac{\partial J_2(\omega, m^2)}{\partial\omega}t_3$$

$$-\omega J_2'(\omega, m^2)t_5 + \omega\Big(t_5 - 2(1 - \cos\theta)t_3\Big)\int_0^1 dx\, J_2'(x\omega, m^2)\Big] - (\omega \to -\omega)$$

$$T_c = -\frac{g^2 e^2}{2m_N f_\pi^2}\tau_3 \omega J_0(\omega, m^2)t_3 - (\omega \to -\omega)$$

$$T_d = \frac{g^2 e^2}{m_N f_\pi^2}\tau_3 \omega \int_0^1 dx\, J_2'(x\omega, m^2)t_3 - (\omega \to -\omega)$$

$$T_e = \frac{g^2 e^2}{2m_N f_\pi^2}(1 - \tau_3)\frac{1}{\omega}\Big(J_2(\omega, m^2) - J_2(0, m^2)\Big)t_3 - (\omega \to -\omega)$$

$$T_f = \frac{g^2 e^2}{4m_N f_\pi^2}\omega\Big(2(\mu_v - \mu_s\tau_3)(t_3 \cos\theta - t_4) + (1 - \tau_3)t_6\Big)\int_0^1 dx(1 - 2x)J_2'(x\omega, m^2) - (\omega \to -\omega)$$

$$T_g = -\frac{g^2 e^2}{4m_N f_\pi^2}\omega\Big(2(\mu_v + \mu_s\tau_3)(t_3 \cos\theta + t_4 - t_5) + (1 + \tau_3)t_6\Big)\int_0^1 dx\, J_2'(x\omega, m^2) - (\omega \to -\omega)$$

$$T_h = -\frac{g^2 e^2}{m_N f_\pi^2}\omega^2 \int_0^1 dy \int_0^{1-x} dx\Big[\Big((7x - 1)(t_6 - t_5) + 7(1 - x - y)t_4\Big)\frac{\partial J_6''(\tilde\omega, m^2 - xyt)}{\partial\tilde\omega}$$

$$+\Big(2V(x, y, \theta)(xt_6 - xt_5 + (1 - x - y)t_4)$$

$$-(1 - x - y)(9xy - x - y)t_7\Big)\omega^2\frac{\partial J_2''(\tilde\omega, m^2 - xyt)}{\partial\tilde\omega}$$

$$-xy(1 - x - y)\omega^4 V(x, y, \theta)t_7\frac{\partial J_0''(\tilde\omega, m^2 - xyt)}{\partial\tilde\omega}\Big] - (\omega \to -\omega) \tag{A1}$$

where $\tilde\omega = (1 - x - y)\omega$,

$$J_6(\omega, m^2) = \frac{1}{d + 1}\Big((m^2 - \omega^2)J_2(\omega, m^2) - \frac{\omega m^2}{d}\Delta_\pi\Big), \tag{A2}$$

$J_0(\omega, m^2)$, $J_2(\omega, m^2)$ and $\Delta_\pi$ have their usual meanings, prime denotes differentiation with respect to $m^2$, and

$$V(x, y, \theta) = (2xy - x - y + 1)\cos\theta - x(1 - x) - y(1 - y). \tag{A3}$$

# REFERENCES

[1] F. Low, Phys. Rev. **96** 1428 (1954); M. Gell-Mann and M. Goldberger, Phys. Rev. **96** 1433 (1954).

[2] S. B. Gerasimov, Sov. J. Nucl. Phys. **2** 430 (1966); S. D. Drell and A. C. Hearn, Phys. Rev. Lett. **16** 908 (1966).

[3] I. Karliner, Phys. Rev. **D 7** 2717 (1973);

[4] A. M. Sandorfi, C. S. Whisnant and M. Khandaker, Phys. Rev. **D 50** R6681 (1994).

[5] A. Thomas, Talk at PANIC 99, Uppsala, Sweden, June 1999.

[6] D. Drechsel and G. Krein, Phys. Rev. **D 58** 116009 (1998).

[7] D. Drechsel, G. Krein and O. Hanstein, Phys. Lett. **B 420** 248 (1998).

[8] V. Bernard, N. Kaiser, J. Kambor and U.-G. Meißner, Nucl. Phys. **B 388** 315 (1992).

[9] T. R. Hemmert, B. R. Holstein J. Kambor and G. Knöchlein, Phys. Rev. **D 57** 5746 (1998).

[10] J. Tonnison, A. M. Sandorfi, S. Hoblit and A. M. Nathan, Phys. Rev. Lett. 80 4382 (1998).

[11] X. Ji, C-W. Kao and J. Osborne, Phys. Rev. **D 61** 074003 (2000).

[12] K. B. V. Kumar, J. A. McGovern and M. C. Birse, `hep-ph/9909442`.

[13] V. Bernard, N. Kaiser and U.-G. Meißner, Int. J. Mod. Phys. E **4** 193 (1995).

[14] G. C. Gellas, T. R. Hemmert and U.-G. Meißner, `nucl-th/0002027`.